Let us first integrate w.r.t. x and then w.r.t. y. As shown in the adjoining figure we take an elementary strip parallel to x-axis and the limits for x are clearly from the line $x=y$ to the curve $x=\sqrt{y}$ and then the limits for y will be from $y=0$ to $y=1$.

$$\therefore I = \int_0^1 \left[\int_y^{\sqrt{y}} xy(x+y)\,dx\right]dy = \int_0^1 \left[\frac{x^3}{3}y + \frac{x^2}{2}y^2\right]_{x=y}^{x=\sqrt{y}} dy$$

$$= \int_0^1 \left[\frac{y^2\sqrt{y}}{3} + \frac{y^3}{2} - \left(\frac{y^4}{3}+\frac{y^4}{2}\right)\right]dy$$

$$= \int_0^1 \left[\frac{y^{5/2}}{3} + \frac{y^3}{2} - \frac{5}{6}y^4\right]dy = \left[\frac{y^{7/2}}{3}\cdot\frac{2}{7} + \frac{y^4}{8} - \frac{5}{6}\cdot\frac{y^5}{5}\right]_0^1$$

$$= \frac{2}{21} + \frac{1}{8} - \frac{1}{6} = \frac{16+21-28}{168} = \frac{37-28}{168} = \frac{\cancel{9}\,3}{\cancel{168}\,56}$$

$$= \frac{3}{56} \quad \text{Ans}$$

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or we integrate first along the vertical strip $P'Q'$ and then slide it from AC to BD. Thus the order of integration is immaterial, provided the limits of integration are changed accordingly.

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right]dy = \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x,y)\,dy\right]dx$$

Problems on Double Integrals.

Q① Prove that $\int_1^2\int_3^4 (xy+e^y)\,dy\,dx = \int_3^4\int_1^2 (xy+e^y)\,dx\,dy$

Soln: $L.H.S. = \int_1^2\int_3^4 (xy+e^y)\,dy\,dx = \int_1^2\left[\int_3^4 (xy+e^y)\,dy\right]dx$

$$= \int_1^2 \left[\frac{xy^2}{2} + e^y\right]_3^4 dx = \int_1^2 \left(8x + e^4 - \frac{9}{2}x - e^3\right)dx$$

$$= \int_1^2 \left(\frac{7}{2}x + e^4 - e^3\right)dx = \left[\frac{7x^2}{4} + (e^4-e^3)x\right]_1^2$$

$$= 7 + 2(e^4-e^3) - \frac{7}{4} - (e^4-e^3) = \frac{21}{4} + e^4 - e^3.$$

Now, $R.H.S. = \int_3^4\int_1^2 (xy+e^y)\,dx\,dy = \int_3^4\left[\int_1^2 (xy+e^y)\,dx\right]dy$

$$= \int_3^4 \left[\frac{yx^2}{2} + xe^y\right]_1^2 dy = \int_3^4 \left(2y + 2e^y - \frac{y}{2} - e^y\right)dy$$

$$= \int_3^4 \left(\frac{3y}{2} + e^y\right)dy = \left[\frac{3y^2}{4} + e^y\right]_3^4 = 12 + e^4 - \frac{27}{4} - e^3$$

$$= \frac{21}{4} + e^4 - e^3 \quad \text{Proved.}$$

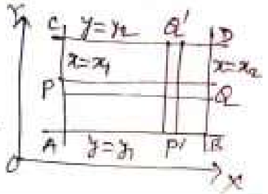
$\iint_R f(x,y)\,dx\,dy$ — limits $x = x_1$ and $x = x_2$ ... function of x is integrated. Thus

$$\int_{x_1}^{x_2}\left[\int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)} f(x,y)\,dy\right]dx.$$

the integration being carried from the inner to the outer rectangle.

(iii) When x_1, x_2, y_1, y_2 are constants.

Here the region of integration R is the rectangle ABCD. We integrate first along the horizontal strip PQ and then slide it from AB to CD;



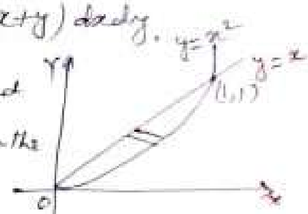
$$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\left[\sin^{-1}x\right]_0^1 dy = \int_0^1 \frac{1}{\sqrt{1-y^2}}\cdot\frac{\pi}{2}\,dy$$

$$= \frac{\pi}{2}\left[\sin^{-1}y\right]_0^1 = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4} \quad \text{Ans}$$

Q(3) Evaluate $\iint xy(x+y)\,dx\,dy$ over the region bounded by the line $y = x$ and the curve $y = x^2$

Sol^n: The region of integration is as shown in the adjoining figure. Let $I = \iint xy(x+y)\,dx\,dy$.

Let us first integrate w.r.t. x and then with w.r.t. y. As shown in the adjoining figure we take an elementary strip parallel to x-axis and the limits for x are clearly from the line $x = y$ to the curve $x = \sqrt{y}$ and then the limits for y will be from $y = 0$ to $y = 1$.



$$\therefore I = \int_0^1\int_y^{\sqrt{y}}\left[xy(x+y)\,dx\right]dy = \int_0^1\left[\frac{x^3}{3}y + \frac{x^2}{2}y^2\right]_{x=y}^{x=\sqrt{y}} dy$$

$$= \int_0^1\left[\frac{y^2\sqrt{y}}{3} + \frac{y^3}{2} - \left(\frac{y^4}{3} + \frac{y^4}{2}\right)\right]dy$$

$$= \int_0^1\left[\frac{y^{5/2}}{3} + \frac{y^3}{2} - \frac{5}{6}y^4\right]dy = \left[\frac{y^{7/2}}{3}\cdot\frac{2}{7} + \frac{y^4}{8} - \frac{5}{6}\cdot\frac{y^5}{5}\right]_0^1$$

$$= \frac{2}{21} + \frac{1}{8} - \frac{1}{6} = \frac{16+21-28}{168} = \frac{37-28}{168} = \frac{\cancel{9}\,3}{\cancel{168}\,56}$$

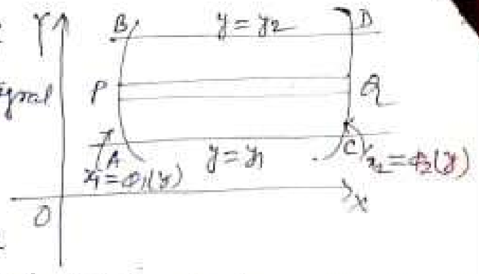
$$= \frac{3}{56} \quad \text{Ans}$$

or we integrate first along the vertical strip P'Q' and then slide it from AC to BD. Thus the order of integration is immaterial, provided the limits of integration are changed accordingly.

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right]dy = \int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x,y)\,dy\right]dx$$

(i) When x_1, x_2 are functions of y and y_1, y_2 are constants.
Let AB and CD be the curves $x_1 = \phi_1(y)$ and $x_2 = \phi_2(y)$.

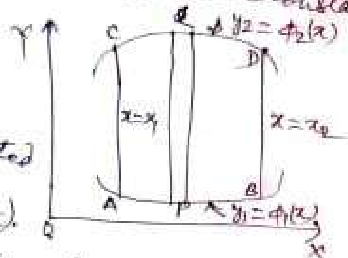
Take a horizontal strip PQ of width δy. Here the double integral is evaluated first w.r.t. x (treating y as a constant).
The resulting expression which is a function of y is integrated w.r.t. y between the limits $y = y_1$ and $y = y_2$. Thus



$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left[\int_{x_1=\phi_1(y)}^{x_2=\phi_2(y)} f(x,y)\,dx\right]dy$$

the integration being carried from the inner to the outer rectangle.

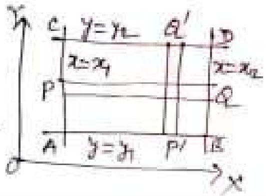
(ii) When y_1, y_2 are functions of x and x_1, x_2 are constants.
Let AB and CD be the curves $y_1 = \phi_1(x)$ and $y_2 = \phi_2(x)$. Take a vertical strip PQ of width δx.
Here the double integral is evaluated first w.r.t. y (treating x as constant).
The resulting expression which is a function of x is integrated w.r.t. x between the limits $x = x_1$ and $x = x_2$. Thus



$$\iint_R f(x,y)\,dx\,dy = \int_{x_1}^{x_2}\left[\int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)} f(x,y)\,dy\right]dx.$$

the integration being carried from the inner to the outer rectangle.

(iii) When x_1, x_2, y_1, y_2 are constants.
Here the region of integration R is the rectangle ABCD. We integrate first along the horizontal strip PQ and then slide it from AB to CD;

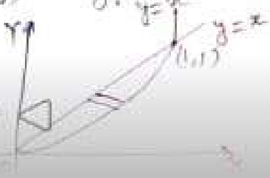


$$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\left[\sin^{-1}x\right]_0^1 dy = \int_0^1 \frac{1}{\sqrt{1-y^2}}\cdot\frac{\pi}{2}\,dy$$

$$= \frac{\pi}{2}\left[\sin^{-1}y\right]_0^1 = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4} \quad \text{Ans}$$

Q③ Evaluate $\iint xy(x+y)\,dx\,dy$ over the region bounded by the line $y = x$ and the curve $y = x^2$.

Sol? The region of integration is as shown in the adjoining figure. Let $I = \iint xy(x+y)\,dx\,dy$.
Let us first integrate w.r.t. x and then with w.r.t. y. As shown in the adjoining figure we take an



Multivariable Calculus: Integration

Multiple Integration: Double and Triple integrals

* **Double Integral:–** Let $f(x,y)$ be a continuous function of x and y, defined within and on a Region R bounded by a Closed Curve C in the xy-plane. Let the region R be subdivided into n subregions of areas $\delta A_1, \delta A_2, \ldots, \delta A_n$. Let (x_r, y_r) be any point within the r^{th} subregion δA_r.

Consider the sum $f(x_1,y_1)\,\delta A_1 + f(x_2,y_2)\,\delta A_2 + \ldots + f(x_n,y_n)\,\delta A_n$

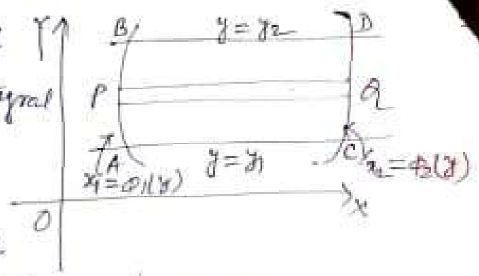
$$= \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \quad \text{—— (1)}$$

The limit of this sum as $n \to \infty$ and $\delta A_r \to 0$ $(r = 1, 2, \ldots, n)$ is defined as the double integral of $f(x,y)$ over the region R and is written as $\iint_R f(x,y)\,dA$.

Which is also expressed as $\iint_R f(x,y)\,dx\,dy$ or $\iint_R f(x,y)\,dy\,dx$

* **Evaluation of Double integrals:** The methods of evaluating the double integrals depend upon the nature of the curves bounding the region R. Let the region R be bounded by the curves $x = x_1$; $x = x_2$ and $y = y_1$, $y = y_2$.

(i) When x_1, x_2 are functions of y and y_1, y_2 are constants.
→ Let AB and CD be the curves $x_1 = \phi_1(y)$ and $x_2 = \phi_2(y)$.



Take a horizontal strip PQ of width δy. Here the double integral is evaluated first w.r.t. x (treating y as a constant).
The resulting expression which is a function of y is integrated w.r.t. y between the limits $y = y_1$ and $y = y_2$. Thus

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2} \left[\int_{x_1=\phi_1(y)}^{x_2=\phi_2(y)} f(x,y)\,dx \right] dy$$

the integration being carried from the inner to the outer rectangle.

(ii) When y_1, y_2 are functions of x and x_1, x_2 are constants.
Let AB and CD be the curves $y_1 = \phi_1(x)$ and $y_2 = \ldots$